TURÁN-TYPE INEQUALITIES FOR GENERALIZED K-BESSEL FUNCTIONS

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Abstract. The present paper proposes an approach to the generalized k-Bessel function defined by

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)=\sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}},$$

where $\mathrm{k}>0$ and $p,q,r\in\mathbb{C}$. We discuss the uniform convergence of $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$. Moreover, we prove that the prescribed function is entire and find its growth order and type. We also find its Weierstrass factorization which turns out to be an infinite product that it is uniformly convergent on a compact subset of the complex plane. The integral representation for $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ is found using the representation for the k-beta functions. The present paper further proves that the specified function is a solution of a second-order differential equation that generalizes certain well-known differential equations for the classical Bessel functions. In addition, some interesting properties like the recurrence as well as differential relations are demonstrated. Some of them may be useful to establish some turán-type inequalities for it. Ultimately, we study the monotonicity and log-convexity of the normalized form of the modified k-Bessel function $\mathrm{T}^{\mathrm{k}}_{p,q,1}$ defined by $\mathrm{T}^{\mathrm{k}}_{p,q,1}(z)=i^{-\frac{p}{k}}\,\mathrm{U}^{\mathrm{k}}_{p,q,1}(iz)$ as well as the quotient of the modified k-Bessel function, exponential and k-hypergeometric functions where the leading concept of the proofs comes from the monotonicity of the ratio of two power series. Further investigations on the prescribed function are underway and will be presented in forthcoming article.

1. Introduction and Preliminaries

In recent years, considerable attention has been given to the role of *Special functions,* abbreviated SFs, especially in mathematics and physics due to their numerous applications. They have long served as an influential tool to use in solving different types of ordinary and partial differential equations. Many mathematicians have recently paid special attention to study some properties of the k-analog of some Special

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functions. Such an extension has been sought to get interesting results analogous to those in the theory of classical functions. For a complete list of references on this subject, interesting readers may consider [14, 16] for k-gamma and k-beta functions, [2, 3, 4, 5, 18, 19, 20, 30, 27] and [36] for k-Bessel functions and [15] for k-zeta and k-hypergeometric functions. Results of this type seem to be quite essential in the development of the theory of Special functions.

Modified k-Bessel functions are closely related to modified Bessel functions which lie among the most essential Special function. Their properties can be useful in a variety of mathematical physics. Applications of *modified Bessel function* can be found in several problems which naturally arise in fluid mechanics, wave mechanics, biophysics, quantum billiards, electrical engineering, finite elasticity, mathematical physics, special relativity, probability and statistics among many others. For more articles on the subject, we refer, e.g., to [8, 21, 22, 23, 28, 34] and the references therein. In this paper, we restrict ourselves to the generalized k-Bessel function. We get some promising results that could be helpful for other research projects on Special functions. Further work regarding the generalized k-Bessel function is underway and will be presented in a future paper.

Turán's inequality for the Legendre polynomials, denoted by $\mathrm{P}_n(x)$, is given by $(\mathrm{P}_n(x))^2 \geq \mathrm{P}_{n+1}(x)\mathrm{P}_{n-1}(x)$, for $x \in [-1,1]$, $n \in \mathbb{N}$, where equality gets valid if $x = \pm 1$ only. This inequality was proved by Szegö [31] and Turán [33]. There are many generalizations of this inequality for various Special functions and orthogonal polynomials. For recent articles regarding this inequality, we refer the reader to [12, 9, 13, 26, 25, 31, 33] and the references therein.

This paper is structured as follows. The uniform convergence of the k-analog of the generalized Bessel function of the first kind of order p, denoted by $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ is discussed. Moreover, it has been proved that the prescribed function is entire and find its growth order, type and Weierstrass factorization. Section 2 is devoted to finding the integral representation for $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ using the representation for the k-beta functions. It has been further proved that the specified function is a solution of a second-order differential equation that generalizes certain well-known differential equations for the classical Bessel functions. Furthermore, some interesting properties like the recurrence as well as differential relations are obtained. Some of them may be useful to establish some Turán type inequalities for it. Ultimately, the monotonicity and log-convexity of the normalized form of the modified k-Bessel function $\mathrm{T}^{\mathrm{k}}_{p,q,1}$ defined by

$\mathrm{T}^{\mathrm{k}}_{p,q,1}(z) = i^{-\frac{p}{k}}\mathrm{U}^{\mathrm{k}}_{p,q,1}(iz)$ are studied as well as the quotient of modified k-Bessel function and the exponential and k-hypergeometric functions where the leading concept of the proofs comes from the monotonicity of the ratio of two power series.

We start this section by recalling the second-order differential equation [35, p. 38]

$$z^2\omega''(z) + z\omega'(z) - (p^2 - z^2)\omega(z) = 0, \tag{1}$$

where p is an unrestricted real or complex number. The homogeneous equation associated with (1) is called Bessel equation. A particular solution of (1) is called Bessel function of the first kind of order p, and it can be expressed as

$$\mathrm{J}_p(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+p+1)\Gamma(n+1)}\left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}.$$

Increasing attention is being given to scientific and technical problems, which lead to differential equation of the modified Bessel type. The modified Bessel equation is given by [35, p. 77]

$$z^2\omega''(z) + z\omega'(z) - (p^2 + z^2)\omega(z) = 0, \tag{2}$$

and a particular solution of this equation is the modified Bessel function of the first kind defined by

$$\mathrm{I}_p(z) := \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+p+1)\Gamma(n+1)}\left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}.$$

In addition, the differential equation

$$z^2\omega''(z) + z\omega'(z) - [p(p+1) - z^2]\omega(z) = 0, \tag{3}$$

which differs from (1) and (2) in the coefficients of $zw'(z)$ and $w(z)$, is called spherical Bessel function. Its solution is called spherical Bessel function of the first kind of order p, and is defined by

$$\mathrm{S}_p(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\left(n+p+\frac{3}{2}\right)\Gamma(n+1)}\left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}.$$

On the other hand, Baricz in [7] has investigated the second-order differential equation

$$z^2\omega''(z) + qz\omega'(z) + [(1-q)p + rz^2 - p^2]\omega(z) = 0, \tag{4}$$

for $p, q, c \in \mathbb{C}$, which can be considered as a solution of

$$\mathrm{U}_{p,q,r}(z) := \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma\left(n+p+\frac{q+1}{2}\right)\Gamma(n+1)}\left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}.$$

Another generalization of the Bessel functions has been established by Mondal and Akel in [27]. This generalization is given by the infinite series representation, for $p > -1,\ \mathrm{k} > 0$ and $c \in \mathbb{R}$,

$$\mathrm{U}_{p,r}^{\mathrm{k}}(z) := \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma_k\left(n\mathrm{k}+p+\mathrm{k}\right)\Gamma\left(n+1\right)} \left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}} \quad \text{for all } z \in \mathbb{C}, \tag{5}$$

which is a particular solution of the following differential equation

$$z^2\omega''(z) + z\omega'(z) - \frac{1}{\mathrm{k}^2}\left(p^2 - rz^2\mathrm{k}\right)\omega(z) = 0. \tag{6}$$

Here Γ_{k} stands for the k-gamma function defined by

$$\Gamma_{\mathrm{k}}(z) := \int_0^{\infty} \ell^{z-1} e^{-\frac{\ell^{\mathrm{k}}}{\mathrm{k}}} d\ell,$$

where $\Re\{z\} > 0$ and $\mathrm{k} > 0$. For more information regarding the k-gamma function, the interested reader is referred to [14, 15, 16] and the references therein. It shall be noted that the classical gamma function and the k-gamma function, for a complex number z and $\mathrm{k} > 0$, are related by $\Gamma_{\mathrm{k}}(z) = \mathrm{k}^{(z/\mathrm{k})-1}\Gamma_{\mathrm{k}}(z/\mathrm{k})$. It is notable that the $\Gamma_{\mathrm{k}}(z)$ satisfies

$$\Gamma_{\mathrm{k}}(z+\mathrm{k}) = z\Gamma_{\mathrm{k}}(z),\ \Gamma_{\mathrm{k}}(z) = 1,\ \frac{1}{\Gamma_{\mathrm{k}}(z)} = z\mathrm{k}^{-\frac{z}{\mathrm{k}}} e^{\frac{z}{\mathrm{k}}\gamma} \prod_{n=1}^{\infty}\left(1+\frac{z}{n\mathrm{k}}\right)e^{-\frac{z}{n\mathrm{k}}},$$

with Euler's constant γ.

It is worth mentioning that as $\mathrm{k} \to 1$ in (5), we obtain the classical Bessel functions $\mathrm{J}_p(z)$ while the classical modified Bessel functions $\mathrm{I}_p(z)$ will follow if we let $\mathrm{k} \to 1$ and $r = -1$.

Recently, in [15], the authors defined the k-hypergeometric function by

$${}_2\mathrm{F}_{1,\mathrm{k}}[(\lambda,\mathrm{k}),(\mu,\mathrm{k});(\nu,\mathrm{k});z] := \sum_{n=0}^{\infty} \frac{(\lambda)_{n,\mathrm{k}}(\mu)_{n,\mathrm{k}}}{(\nu)_{n,\mathrm{k}}} \frac{z^n}{n!}, \tag{7}$$

which is a particular solution of the following differential equation

$$\mathrm{k}z(1-\mathrm{k}z)\omega''(z) - \left[(\lambda+\mu+\mathrm{k})z - \gamma\right] z\omega'(z) - \lambda\mu\omega(z) = 0,$$

where $(x)_{n,\mathrm{k}}$ stands for the Pochhammer k-symbol defined by

$$(x)_{n,\mathrm{k}} := \frac{\Gamma_{\mathrm{k}}(x+nk)}{\Gamma_{\mathrm{k}}(x)} = x(x+\mathrm{k})(x+2\mathrm{k})...(x+(n-1)\mathrm{k}).$$

We proceed to recall that a function U that is real entire function belongs to the Laguerre-Pólya class $\mathcal{LP}$ if it can be written as follows

$$\mathrm{U}(z) = \mathrm{c}x^m e^{-\mathrm{a}x^2+\mathrm{b}x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{x_n}\right) e^{-\frac{x}{x_n}},$$

where $\mathrm{c}, \mathrm{b}, x_n \in \mathbb{R},\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\sum x_n^{-2} < \infty$. The class $\mathcal{LP}$ consists of entire functions which can be approximated by polynomials with only real zeros, uniformly on the compact sets of the complex plane and it is closed under differentiation.

The main object of this paper is to introduce an elegant power series $\mathrm{W}^{\mathrm{k}}_{p,b,c}(z)$ defined by

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z) := \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + p + \frac{q+1}{2}\mathrm{k}\right)\Gamma\left(n+1\right)} \left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}, \tag{8}$$

where $\mathrm{k} > 0$ and $p, q, r \in \mathbb{C}$.

Using the power series (8), it is a considerable computational and conceptual advantage for the prescribed function to provide explicit and interpretable solutions for problems in the forthcoming sections.

Before we proceed, we will present the following definitions and lemmas that will be used in the proofs of the main results of this paper.

Definition 1. Let X be a nonempty convex set in $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is convex on X if one of the following conditions hold:
(i) if $\alpha f(x) + (1-\alpha)f(y) \geq f(\alpha x + (1-\alpha)y)$ for every $\alpha \in [0,1]$ and $x, y \in X$;
(ii) if f is increasing, twice differentiable and f'' is non-negative.

Definition 2. Let X be a nonempty convex set in $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is
(i) concave on X if $-f$ is convex on X;
(ii) strictly convex on X if the inequality in (i) in Definition 1 is $<$ and strictly concave on X if $-f$ is strictly convex on X;
(iii) log-convex or superconvex on X if $f > 0$ and $\log f$ is convex on X, that is, $\alpha \log f(x) + (1-\alpha)\log f(y) \geq \log f(\alpha x + (1-\alpha)y)$, or equivalently, $(f(x))^{\alpha}(f(y))^{1-\alpha} \geq f(\alpha x + (1-\alpha)y)$, for every $x, y \in X$ and $\alpha \in [0,1]$ and log-concave if the $-f$ is log-convex;
(iv) absolutely monotonic on X if it has derivatives of all orders and satisfies $f^{(n)}(x) \geq 0$ for every $x \in X$ and $n \geq 0$.

Lemma 1. [11]. *Let* $a_n \in \mathbb{R}$ *and* $b_n > 0$ *for* $n \in \mathbb{N}_0$. *If* $A(z) = \sum\limits_{n\geq 0} a_n z^n$ *and* $B(z) = \sum\limits_{n\geq 0} b_n z^n$ *be a convergent power series in* $|z| < R$ *and* $\{a_n/b_n\}_{n\geq 0}$ *is (strictly)* *increasing (decreasing), then the ratio* $A(z)/B(z)$ *is (strictly) increasing (decreasing) on* $(0,R)$.

Note that the above lemma can be applied if $A(z)$ and $B(z)$ are power series of the form

$$A(z)=\sum_{n\geq 0} a_n z^{2n} \text{ and } B(z)=\sum_{n\geq 0} b_n z^{2n}.$$

Lemma 2. [17] *Let* $f:(0,\infty)\to\mathbb{R}$ *be a function. If* $f(z)/z$ *is increasing, then* f *is super-additive.*

Lemma 3. [29] *Suppose that* $f(z)=\sum_{n\geq 0} a_n z^n$ *and that can be represented by* $f(z)=e^{\mathrm{a}z^2}h(z),\ a\leq 0,$ *and* h *is of the form*

$$h(z)=\alpha e^{\beta z}\prod_{n=1}^{\infty}\left(1-\frac{z}{c_n}\right)e^{-\frac{z}{c_n}},$$

with $\alpha,\beta\in\mathbb{R},\ \sum_{n\geq 1}\left|c_n\right|^{-2}<\infty,\ f$ *has real zeros (or no zeros at all), and* $g(n)$ *is of the form*

$$g(z)=e^{\delta z}\prod_{n=1}^{\infty}\left(1+\frac{z}{d_n}\right)e^{-\frac{x}{d_n}},$$

where $\delta\in\mathbb{R},\ d_n>0,\ \sum_{n\geq 1} d_n^{-2}<\infty,$ *then* $\sum_{n\geq 0} a_n g(n) z^n$ *has only real zeros.*

Lemma 4. [24] *Let* f *be an entire function with infinite zeros* $0,a_1,a_2,...,a_n,...,$ *then* f *can be represented as*

$$f(z)=z^m e^{g(z)}\prod_{n=1}^{\infty}G\left(\frac{z}{a_n};p\right),$$

where

$$G\left(w,p\right)=\begin{cases} 1-w & \text{if } p=0,\\ (1-w)\exp(w+\frac{w^2}{2}+...+\frac{w^p}{p}) & \text{if } p>0,\end{cases}$$

and m *is the order of zero of* f *at* $z=0,\ g$ *is an entire function.*

Lemma 5. *Suppose that* $p>0,\ \mathrm{k}>0$ *and* $(p/\mathrm{k})+(q-1)/2>0,$ *then the function* $\mathrm{U}_{p,b,r}^{\mathrm{k}}$ *is an entire function of order* $\rho=1/2$ *and type* $\tau=2\sqrt{r}/\sqrt{\mathrm{k}}$.

Proof. First of all, we shall prove that (8) is uniformly convergent on a bounded subset of the complex plane. Noting that

$$\begin{aligned}&\left|\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\right|\\ &=\left|\frac{(-r)^n}{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\right|\\ &\leq\frac{|r|^n}{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}\frac{|z|^{2n+\frac{p}{\mathrm{k}}}}{2^{2n+\frac{p}{\mathrm{k}}}}.\end{aligned}$$

Using the D'Alembert's ratio test, we have

$$\lim_{n\to\infty}\frac{|r|\,|z|^2}{4\mathrm{k}(n+1)\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}=0,$$

so, by using the Weierstrass test, the uniform convergence follows. On the other hand, (8) has an infinite radius of convergence as follows

$$\begin{aligned}\frac{1}{R}&=\limsup_{n\to\infty}\left|\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\right|^{1/n}\\ &=\limsup_{n\to\infty}\left|\frac{(-r)^n}{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}\right|^{1/n}=0,\end{aligned}$$

and so, it represents an entire function. In addition, the order of the entire function $\mathrm{U}^{\mathrm{k}}_{p,b,r}$ can be evaluated as

$$\begin{aligned}\rho&=\limsup_{n\to\infty}\frac{n\log n}{\log\left|\dfrac{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{b-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{b+1}{2}\right)n!}{(-r)^n}\right|}\\ &=\limsup_{n\to\infty}\frac{n\log n}{\log\left[\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)\right]+\log\Gamma(n+1)-n\log r}=\frac{1}{2},\end{aligned}$$

and the type of $\mathrm{U}^{\mathrm{k}}_{p,b,r}$ is given by

$$\tau=\frac{1}{\rho e}\limsup_{n\to\infty}\left\{n\left|\frac{(-r)^n}{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}\right|^{1/2n}\right\}=\frac{2\sqrt{r}}{\sqrt{\mathrm{k}}},$$

which ends the proof. □

The proof of the next lemma goes in the same steps as [32, Lemma 2.1]

Lemma 6. *Assume that* $p, r, \mathrm{k} > 0$ *such that* $(p/\mathrm{k}) + (q-1)/2 > 0$, *then* $z \mapsto \mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ *has infinite number of zeros and these zeros are real. Further, the Weierstrass factorization of* $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ *is*

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z) = \frac{\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}}{\Gamma_{\mathrm{k}}\left(p + \frac{q+1}{2}\mathrm{k}\right)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\left(\mathrm{u}^{\mathrm{k}}_{p,q,r,n}\right)^2}\right), \tag{9}$$

where $\mathrm{u}^{\mathrm{k}}_{p,q,r,n}$ *stands for the* n*th positive zero of* $\mathrm{U}^{\mathrm{k}}_{p,q,r}$ *and the uniform convergence of the above infinite product on a compact subset of the complex plane holds.*

Proof. Lemma (3) is applied to prove that the zeros of $\mathrm{U}^{\mathrm{k}}_{p,q,r}$ are real. This will be done by setting $f(z) = e^{-r(z/2)^2}$ and $g : [0, \infty) \to \mathbb{R}$ with

$$g(n) = \frac{1}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + p + \frac{q+1}{2}\mathrm{k}\right)}.$$

It is notable that f is in the Laguerre-Pólya class $\mathcal{LP}$ as well as g. So, by Lemma (3), the zeros of $\mathrm{U}^{\mathrm{k}}_{p,q,r}$ are real if $p, r, \mathrm{k} > 0$ and $(p/\mathrm{k}) + (q-1)/2 > 0$.

Now, we will recall that $\mathrm{U}^{\mathrm{k}}_{p,q,r}$ is of order $1/2$ from Lemma (5) and by making use of the Hadamard factorization theorem, every function of non-integral order has infinitely many zeros. It is concluded that $\mathrm{U}^{\mathrm{k}}_{p,q,r}$ has infinitely many zeros and all are real. Thanks to (4), the infinite product (9) follows. □

Now that we have recalled and developed the necessary basics, we can move on to the main results of this paper.

2. PROPERTIES OF THE GENERALIZED k-BESSEL FUNCTIONS

In the following, the integral representation of the generalized k-Bessel functions is established based upon the next representation for the k-beta functions (see [15])

$$\mathrm{B}_{\mathrm{k}}(\alpha, \beta) = \frac{\Gamma_{\mathrm{k}}(\alpha)\Gamma_{\mathrm{k}}(\beta)}{\Gamma_{\mathrm{k}}(\alpha+\beta)} = \frac{1}{\mathrm{k}} \int_0^1 \ell^{\frac{\alpha}{\mathrm{k}}-1}(1-\ell)^{\frac{\beta}{\mathrm{k}}-1} d\ell.$$

Note that if we now replace ℓ by ℓ^2 on the above integral, then we obtain

$$\mathrm{B}_{\mathrm{k}}(\alpha, \beta) = \frac{2}{\mathrm{k}} \int_0^1 \ell^{\frac{2\alpha}{\mathrm{k}}-1}(1-\ell^2)^{\frac{\beta}{\mathrm{k}}-1} d\ell, \tag{10}$$

with $\Re\{\alpha\} > 0$, $\Re\{\beta\} > 0$ and $\mathrm{k} > 0$. For $\alpha = n\mathrm{k} + \frac{q+1}{2}\mathrm{k}$ and $\beta = p$, it finds

$$\mathrm{B}_{\mathrm{k}}\left(n\mathrm{k} + p + \tfrac{q+1}{2}\mathrm{k}, p\right) = \frac{\Gamma_{\mathrm{k}}(n\mathrm{k} + \frac{q+1}{2}\mathrm{k})\Gamma_{\mathrm{k}}(p)}{\Gamma_{\mathrm{k}}(n\mathrm{k} + p + \frac{q+1}{2}\mathrm{k})} = \frac{2}{\mathrm{k}} \int_0^1 \ell^{2n+q}(1-\ell^2)^{\frac{p}{\mathrm{k}}-1} d\ell,$$

where $\Re\{p\} > 0,\ \Re\{q\} > -1$ and $\mathrm{k} > 0$.

It is significant to note that (8) can be expressed as

$$\begin{aligned}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z) &= \sum_{n=0}^{\infty} \frac{(-r)^n \mathrm{B}_{\mathrm{k}}\left(n\mathrm{k} + p + \frac{q+1}{2}\mathrm{k}, p\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + \frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}(p)\, n!} \left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}} \\ &= \frac{2}{\mathrm{k}} \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + \frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}(p)\, n!} \left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}} \int_0^1 \ell^{2n+q}(1-\ell^2)^{\frac{p}{\mathrm{k}}-1} d\ell \\ &= \frac{2}{\Gamma_{\mathrm{k}}(p)} \left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}} \int_0^1 \ell^q (1-\ell^2)^{\frac{p}{\mathrm{k}}-1} \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + \frac{q+1}{2}\mathrm{k}\right) n!} \left(\frac{z}{2}\right)^{2n} d\ell.\end{aligned}$$

By making use of the property $\Gamma_{\mathrm{k}}(\mathrm{k}z) = \mathrm{k}^{z-1}\Gamma(z)$, we obtain

$$\begin{aligned}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z) &= \frac{2}{\Gamma_{\mathrm{k}}(p)} \left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}} \int_0^1 \ell^q (1-\ell^2)^{\frac{p}{\mathrm{k}}-1} \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma\left(n + \frac{q+1}{2}\right) n!} \left(\frac{\ell z}{2\sqrt{\mathrm{k}}}\right)^{2n} d\ell \\ &= \frac{2^{\frac{b+1}{2}}}{\mathrm{k}^{\frac{b-1}{2}}\Gamma_{\mathrm{k}}(p)} \left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}} \int_0^1 \ell^q (1-\ell^2)^{\frac{p}{\mathrm{k}}-1} \left(\frac{\sqrt{r}\ell z}{\sqrt{\mathrm{k}}}\right)^{\frac{1-q}{2}} \mathrm{J}_{\frac{q-1}{2}}\left(\frac{\sqrt{r}\ell z}{\sqrt{\mathrm{k}}}\right) d\ell.\end{aligned}$$

Another integral representation for the $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ will follow by setting $\alpha = n\mathrm{k} + \frac{\mathrm{k}}{2}$ and $\beta = p + \frac{q}{2}\mathrm{k}$ in (10) as outlined below. From (10), we have

$$\mathrm{B}_{\mathrm{k}}\left(n\mathrm{k} + \frac{\mathrm{k}}{2}, p + \frac{b}{2}\mathrm{k}\right) = \frac{\Gamma_{\mathrm{k}}(n\mathrm{k} + \frac{\mathrm{k}}{2})\Gamma_{\mathrm{k}}(p + \frac{b}{2}\mathrm{k})}{\Gamma_{\mathrm{k}}(n\mathrm{k} + p + \frac{b+1}{2}\mathrm{k})} = \frac{2}{\mathrm{k}}\int_0^1 \ell^{2n}(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{b}{2}-1} d\ell,$$

Now from (8), we have

$$\begin{aligned}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z) &= \frac{2}{\mathrm{k}} \sum_{n=0}^{\infty} \frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + \frac{\mathrm{k}}{2}\right)\Gamma_{\mathrm{k}}\left(p + \frac{q}{2}\mathrm{k}\right) n!} \left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}} \int_0^1 \ell^{2n}(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1} d\ell \\ &= \frac{2}{\mathrm{k}^{\frac{1}{2}}\Gamma_{\mathrm{k}}\left(p + \frac{q}{2}\mathrm{k}\right)} \left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}} \int_0^1 (1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1} \sum_{n=0}^{\infty} \frac{(-r)^n \ell^{2n}}{\mathrm{k}^n \Gamma\left(n + \frac{1}{2}\right) n!} \left(\frac{z}{2}\right)^{2n} d\ell.\end{aligned}$$

Using the *Legendre's duplication formula*,

$$\Gamma(z)\,\Gamma\left(z + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2z-1}}\Gamma(2z),$$

for $z = n + 1/2$ we get

$$\Gamma\left(n + \frac{1}{2}\right)\Gamma(n+1) = 2^{-2n}\sqrt{\pi}\,\Gamma(2n+1),$$

and so,

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)=\frac{2}{\mathrm{k}^{\frac{1}{2}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}\int_0^1(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1}\sum_{n=0}^{\infty}\frac{(-r)^n\ell^{2n}}{\mathrm{k}^n2^{-2n}\sqrt{\pi}\Gamma(2n+1)}\left(\frac{z}{2}\right)^{2n}d\ell$$

$$=\frac{2}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}\int_0^1(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1}\sum_{n=0}^{\infty}\frac{(-r)^n\left(\ell z\right)^{2n}}{\mathrm{k}^n\Gamma(2n+1)}d\ell. \tag{11}$$

If $r>0$, then (11) leads to

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)=\frac{2}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}\int_0^1(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1}\cos\left(\frac{\sqrt{r}\ell z}{\sqrt{\mathrm{k}}}\right)d\ell. \tag{12}$$

If $r<0$, (11) yields

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)=\frac{2}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}\int_0^1(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1}\cosh\left(\frac{\sqrt{r}\ell z}{\sqrt{\mathrm{k}}}\right)d\ell. \tag{13}$$

Substituting $\ell=\cos t$ in (12) and (13), we get

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)=\frac{-2}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}\int_{\frac{\pi}{2}}^{0}(1-\cos^2t)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1}\cos\left(\frac{\sqrt{r}z\cos t}{\sqrt{\mathrm{k}}}\right)\sin t dt$$

$$=\frac{2}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}\int_0^{\frac{\pi}{2}}(\sin t)^{\frac{2p}{\mathrm{k}}+q-1}\cos\left(\frac{\sqrt{r}z\cos t}{\sqrt{\mathrm{k}}}\right)dt, \tag{14}$$

and

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)=\frac{2}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}\int_0^{\frac{\pi}{2}}(\sin t)^{\frac{2p}{\mathrm{k}}+q-1}\cosh\left(\frac{\sqrt{r}z\cos t}{\sqrt{\mathrm{k}}}\right)dt. \tag{15}$$

In the above integral representations (14) and (15), if we consider as a special cases $\mathrm{k}\to 1$ and $q=r=1$, the well-known representations for the Bessel and the modified Bessel functions are obtained (see [1]).

We now move on to another result of this section. In particular, we prove that $\mathrm{U}^{\mathrm{k}}_{p,q,r}$ is a solution of a second-order differential equation which generalizes certain well-known differential equations for the classical Bessel functions as in the following proposition.

Proposition 1. *Suppose that* $p,q,r\in\mathbb{C}$, $\mathrm{k}>0$ *and* $p+\frac{q+1}{2}\mathrm{k}\neq 0,-\mathrm{k},-2\mathrm{k},...,$ *then* $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ *satisfies the second-order differential equation*

$$z^2\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)''+qz\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'+\frac{1}{\mathrm{k}^2}[r\mathrm{k}z^2-p^2-p\mathrm{k}(q-1)]\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)=0. \tag{16}$$

Remark 1. The above proposition generalizes the results given in [7] and [27] when $\mathrm{k} \to 1$ and $q = 1$, respectively.

In the remainder of this section, the recurrence as well as differential relations for generalized k-Bessel function are obtained as in the following propositions.

Proposition 2. *Assume that* $p, q, r \in \mathbb{C},\ \mathrm{k} > 0$ *and* $p + \frac{q+1}{2}\mathrm{k} \neq 0, -\mathrm{k}, -2\mathrm{k}, ...,$ *then*
(i) $z\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z) + \mathrm{k}rz\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z) = [2p + (q-1)\mathrm{k}]\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$;
(ii) $\mathrm{k}z\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)' + [p + (q-1)\mathrm{k}]\mathrm{U}^{\mathrm{k}}_{p,q,r}(z) = z\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)$;
(iii) $\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z) = \frac{(q-1)\mathrm{k}}{z} + \sum\limits_{m=0}^{\infty}(p + 2m\mathrm{k})\mathrm{U}^{\mathrm{k}}_{p+2m\mathrm{k},q,r}(z)$;
(iv) $\mathrm{k}z\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)' + \mathrm{k}cz\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z) = p\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$;
(v) $2\mathrm{k}z\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)' = z\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z) - \mathrm{k}cz\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z) + (1-q)\mathrm{k}z\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$.

The above proposition generalizes some of the recurrence relations given in [7, Proposition 2.14] and [27] when $\mathrm{k} \to 1$ and $q = 1$, respectively. On the other hand, (i) and (iv) for $q = r = 1$ among other relations have been provd for k-Bessel function $\mathrm{J}^{\mathrm{k}}_p(z)$ in [19] using the series expansion of $\mathrm{J}^{\mathrm{k}}_p(z)$. Here, another proof is given using the generating function for $\mathrm{J}^{\mathrm{k}}_p(z)$ which has been obtained in [18]. The generating function for $\mathrm{J}^{\mathrm{k}}_p(z)$ has the form

$$\mathrm{G}_{\mathrm{k}}(x,t) = e^{(z/2\sqrt{\mathrm{k}})((x/\sqrt{\mathrm{k}})-(\sqrt{\mathrm{k}}/x))} = \sum_{p=-\infty}^{\infty} x^p \mathrm{J}^{\mathrm{k}}_{p\mathrm{k}}(z),$$

for all $x \neq 0$ and finite z. Differentiating $\mathrm{G}_{\mathrm{k}}(x,t)$ with respect to z and x yield

$$\frac{\partial \mathrm{G}_{\mathrm{k}}(x,t)}{\partial z} = \frac{1}{2\sqrt{\mathrm{k}}}((x/\sqrt{\mathrm{k}}) - (\sqrt{\mathrm{k}}/x))e^{(z/2\sqrt{\mathrm{k}})((x/\sqrt{\mathrm{k}})-(\sqrt{\mathrm{k}}/x))} = \sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}}\left(\mathrm{J}^{\mathrm{k}}_p(z)\right)',$$

$$\frac{\partial \mathrm{G}_{\mathrm{k}}(x,t)}{\partial x} = \frac{z}{2\sqrt{\mathrm{k}}}((1/\sqrt{\mathrm{k}}) + (\sqrt{\mathrm{k}}/x^2))e^{(z/2\sqrt{\mathrm{k}})((x/\sqrt{\mathrm{k}})-(\sqrt{\mathrm{k}}/x))} = \sum_{p=-\infty}^{\infty} \frac{p}{\mathrm{k}} x^{\frac{p}{\mathrm{k}}-1}\mathrm{J}^{\mathrm{k}}_p(z).$$

Replacing $\mathrm{G}_{\mathrm{k}}(x,t)$ with its power series expansion in the above equation yields

$$\frac{\partial \mathrm{G}_{\mathrm{k}}(x,t)}{\partial z} = \frac{1}{2\sqrt{\mathrm{k}}}((x/\sqrt{\mathrm{k}}) - (\sqrt{\mathrm{k}}/x)) \sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}}\mathrm{J}^{\mathrm{k}}_p(z) = \sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}}\left(\mathrm{J}^{\mathrm{k}}_p(z)\right)',$$

$$\frac{\partial \mathrm{G}_{\mathrm{k}}(x,t)}{\partial x} = \frac{z}{2\sqrt{k}}((1/\sqrt{\mathrm{k}}) + (\sqrt{\mathrm{k}}/x^2)) \sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}}\mathrm{J}^{\mathrm{k}}_p(z) = \sum_{p=-\infty}^{\infty} \frac{p}{\mathrm{k}} x^{\frac{p}{\mathrm{k}}-1}\mathrm{J}^{\mathrm{k}}_p(z).$$

Hence,

$$\sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}}\left(\mathrm{J}_p^{\mathrm{k}}(z)\right)' = \frac{1}{2\mathrm{k}}\sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}}\mathrm{J}_{p-\mathrm{k}}^{\mathrm{k}}(z) - \frac{1}{2}\sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}}\mathrm{J}_{p+\mathrm{k}}^{\mathrm{k}}(z),$$

$$\sum_{p=-\infty}^{\infty} \frac{p}{\mathrm{k}} x^{\frac{p}{\mathrm{k}}-1}\mathrm{J}_p^{\mathrm{k}}(z) = \frac{z}{2\mathrm{k}}\sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}-1}\mathrm{J}_{p-\mathrm{k}}^{\mathrm{k}}(z) + \frac{z}{2}\sum_{p=-\infty}^{\infty} x^{\frac{p}{\mathrm{k}}-1}\mathrm{J}_{p+\mathrm{k}}^{\mathrm{k}}(z),$$

gives the required recurrence relations.

Proposition 3. *If* $p, q, r \in \mathbb{C},\ \mathrm{k} > 0$ *such that* $p + \frac{q-1}{2}\mathrm{k} \neq 0, -\mathrm{k}, -2\mathrm{k}, ...,$ *then the following identities hold:*

$$\frac{d}{dz}\left(z^{\frac{p}{\mathrm{k}}+q-1}\mathrm{U}_{p,q,r}^{\mathrm{k}}(z)\right) = \frac{z^{\frac{p}{\mathrm{k}}+q-1}}{\mathrm{k}}\mathrm{U}_{p-\mathrm{k},q,r}^{\mathrm{k}}(z), \tag{17}$$

$$\frac{d}{dz}\left(z^{-\frac{p}{\mathrm{k}}}\mathrm{U}_{p,q,r}^{\mathrm{k}}(z)\right) = -rz^{-\frac{p}{\mathrm{k}}}\mathrm{U}_{p+\mathrm{k},q,r}^{\mathrm{k}}(z). \tag{18}$$

Proof. By using the series expansion (8), the result follows. ☐

Proposition 4. *For* $p, q, r \in \mathbb{C},\ \mathrm{k} > 0$ *such that* $p - m\mathrm{k} + \frac{q+1}{2}\mathrm{k} \neq 0, -\mathrm{k}, -2\mathrm{k}, ...,$ *the following identities hold:*

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m\left(z^{\frac{p}{\mathrm{k}}+q-1}\mathrm{U}_{p,q,r}^{\mathrm{k}}(z)\right) = \frac{1}{\mathrm{k}^m}z^{\frac{p}{\mathrm{k}}+q-(m+1)}\mathrm{U}_{p-m\mathrm{k},q,r}^{\mathrm{k}}(z), \tag{19}$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m\left(z^{-\frac{p}{\mathrm{k}}}\mathrm{U}_{p,q,r}^{\mathrm{k}}(z)\right) = (-r)^m z^{-\frac{p}{\mathrm{k}}-m}\mathrm{U}_{p+m\mathrm{k},q,r}^{\mathrm{k}}(z). \tag{20}$$

3. TURÁN-TYPE INEQUALITIES AND MONOTONICITY PROPERTIES FOR THE GENERALIZED k-BESSEL FUNCTION

In the next, some Turán-type inequalities are obtained for $\mathrm{U}_{p,q,r}^{\mathrm{k}}(z)$ utilizing the recurrence relations (ii), (iv) and (v) of Lemma (2) as well as the differential equation (16) as outlined below. The following further lemma is needed.

Lemma 7. *If* $\mathrm{k} > 0,\ p, q$ *and* r *are non-negative real numbers sunch that* $p + (q+1)(\mathrm{k}/2) \geq r/4,$ *then* $\mathrm{U}_{p,q,r}^{\mathrm{k}}(z) \geq 0$ *for all* $z \in (0,1)$.

Proof. Assume that

$$R_n(z) = \frac{r^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k} + p + \frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}, \tag{21}$$

so that

$$\mathrm{U}_{p,q,r}^{\mathrm{k}}(z) = \sum_{n=0}^{\infty}(-1)^n R_n(z). \tag{22}$$

Using the fact that $\Gamma_{\mathrm{k}}(\mathrm{k}z) = \mathrm{k}^{\frac{z}{\mathrm{k}}-1}\Gamma(z/\mathrm{k})$, we get

$$\Gamma_{\mathrm{k}}\left(n\mathrm{k} + p + \frac{q+1}{2}\mathrm{k}\right) = \mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n + \frac{p}{\mathrm{k}} + \frac{q+1}{2}\right).$$

Moreover,

$$\begin{aligned} R_n(z) - R_{n+1}(z) &= \frac{r^n\,(z/2)^{2n+\frac{p}{\mathrm{k}}}}{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}\left[1 - \frac{r(z/2)^2}{(n+1)\mathrm{k}\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)}\right] \\ &\geq \frac{r^n\,(z/2)^{2n+\frac{p}{\mathrm{k}}}}{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}\left[1 - \frac{r}{4(n+1)\mathrm{k}\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)}\right] \\ &= \frac{r^n\,(z/2)^{2n+\frac{p}{\mathrm{k}}}}{\mathrm{k}^{n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}}\Gamma\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)n!}\frac{\Phi(n)}{4(n+1)\mathrm{k}\left(n+\frac{p}{\mathrm{k}}+\frac{q+1}{2}\right)}, \end{aligned}$$

where

$$\Phi(n) := 4(n+1)\mathrm{k}\left(n + \frac{p}{\mathrm{k}} + \frac{q+1}{2}\right) - r.$$

Since $n \geq 0$, then $\Phi(n) \geq 0$ for all $n \geq 0$ under the given hypotheses which implies that $R_n(z) \geq R_{n+1}(z)$ for all $n \geq 0$ while $R_n(z)$ tends to zero as $n \to \infty$. On the other hand, $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ can be written as

$$\mathrm{U}^{\mathrm{k}}_{p,q,r}(z) = R_0(z) - R_1(z) + R_2(z) - R_3(z) + ... \geq 0,$$

under the above hypotheses which ends the proof. □

Theorem 1. *The following inequalities hold:*
(i) Let $p, r, \mathrm{k} > 0,\ q, z \in \mathbb{R}$ *and* $(p/\mathrm{k}) + (q-1)/2 > 0$, *then*

$$\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)^2 - \mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z) \geq 0; \tag{23}$$

(ii) If $p, r, \mathrm{k} > 0,\ q, z > 0$ *and* $(p/\mathrm{k}) + (q-1)/2 > 0$, *then the inequality (23) holds.*
(iii) If $p, r, \mathrm{k} > 0,\ z \in (0,1),\ 0 < q < 1$ *such that* $p + (q+1)(\mathrm{k}/2) \geq r/4$, *then the inequality (23) holds.*

Proof. Taking the logarithmic derivative of (9), we get

$$\frac{z\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)} = \frac{p}{\mathrm{k}} + \sum_{n=1}^{\infty}\frac{2z^2}{z^2 - (\mathrm{u}^{\mathrm{k}}_{p,q,r,n})^2}. \tag{24}$$

Using (ii) of lemma (2), this would leads to

$$\frac{\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)} = \frac{2p + (q-1)\mathrm{k}}{z} + \sum_{n=1}^{\infty}\frac{2\mathrm{k}z}{z^2 - (\mathrm{u}^{\mathrm{k}}_{p,q,r,n})^2}.$$

Now,

$$\begin{aligned}
&\frac{\left(\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\right)^2-\left(\mathrm{U}^{\mathrm{k}}_{p-2\mathrm{k},q,r}(z)\right)\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)}{\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)^2}\\
&=\frac{\mathrm{k}}{z}\frac{\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}-\frac{\mathrm{k}\left(\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\right)'}{\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)}\\
&+\frac{\mathrm{k}\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'}{\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)^2}\left[\frac{2p+(q-1)\mathrm{k}}{z}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)+\mathrm{k}\left(\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\right)'\right]\\
&=\frac{\mathrm{k}}{z}\frac{\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}-\frac{\mathrm{k}\left(\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\right)'}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}+\frac{\mathrm{k}\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'}{\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)^2}\\
&=\frac{\mathrm{k}}{z}\frac{\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}-\mathrm{k}\left(\frac{\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}\right)'=\frac{2p}{z^2}+\sum_{n=1}^{\infty}\frac{4\mathrm{k}z^2}{\left[z^2-(\mathrm{u}^{\mathrm{k}}_{p,q,r,n})^2\right]^2}\geq 0,
\end{aligned}$$

if $p, r, \mathrm{k} > 0,\ q, z \in \mathbb{R}$ and $(p/\mathrm{k}) + (q-1)/2 > 0$. Replacing p to $p+\mathrm{k}$ to find (23);
(ii) Here, we shall recall (iv) of Lemma (2) and the second-order differential equation (16) to find

$$\begin{aligned}
&\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)^2-\left(\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\right)\left(\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)\right)\\
&=\frac{1}{\mathrm{k}rz}\left[-\mathrm{k}^2z^2\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'-\mathrm{k}^2qz\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'\right]\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)\\
&+\frac{(q-1)\mathrm{k}}{rz}\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'+\frac{\mathrm{k}}{r}\left[\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'\right]^2\\
&=-\frac{\mathrm{k}}{r}\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)''-\frac{\mathrm{k}}{rz}\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'\\
&+\frac{\mathrm{k}}{r}\left[\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'\right]^2=-\frac{\mathrm{k}}{cz}\left[\frac{z\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'}{\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}\right]\\
&=-\frac{\mathrm{k}}{rz}\sum_{n=1}^{\infty}\frac{4z(\mathrm{u}^{\mathrm{k}}_{p,q,r,n})^2}{\left[z^2-(\mathrm{u}^{\mathrm{k}}_{p,q,r,n})^2\right]^2}\geq 0,
\end{aligned}$$

if $p, r, \mathrm{k} > 0,\ b, z > 0$ and $(p/\mathrm{k}) + (q-1)/2 > 0$, which completes the proof of (ii);
(iii) Suppose that

$$\Delta^{\mathrm{k}}_{p,q,r}(z)=\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)^2-\left(\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\right)\left(\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)\right).$$

By making use of (ii), (v) and (iv) of Proposition (2), we have

$$\left(\Delta^{\mathrm{k}}_{p,q,r}(z)\right)'=\frac{1}{\mathrm{k}z}\left[(q+1)\mathrm{k}\left(\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)\right)\left(\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)\right)+(1-q)\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)^2\right].$$

From Lemma (7), it is easy to verify that $\left(\Delta^{\mathrm{k}}_{p,q,r}(z)\right)' \geq 0$ under the given hypotheses which ends our claim. ☐

Now, the modified k-Bessel function may be introduced as

$$\mathrm{T}^{\mathrm{k}}_{p,q,1}(z) = i^{-\frac{p}{\mathrm{k}}}\mathrm{U}^{\mathrm{k}}_{p,q,1}(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}},$$

and the normalized form of $\mathrm{T}^{\mathrm{k}}_{p,q,1}(z)$ by

$$\mathcal{U}^{\mathrm{k}}_{p,q,1}(z) := \mathcal{U}^{\mathrm{k}}_{p,q}(z) = \left(\frac{z}{2}\right)^{-\frac{p}{\mathrm{k}}}\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)\mathrm{T}^{\mathrm{k}}_{p,q,r}(z) = \sum_{n=0}^{\infty}\frac{\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n}.$$

We are ready now to pass on the main results of this part.

Theorem 2. *Let* $\mathrm{k} > 0$ *and* $p, q \in \mathbb{R}$*. Then the following assertions hold:*
(i) If $p_1 \geq p > -\frac{\mathrm{k}}{2}(q+1)$*, then* $z \mapsto \mathcal{U}^{\mathrm{k}}_{p,q}(z)/\mathcal{U}^{\mathrm{k}}_{p_1,q}(z)$ *is increasing on* $(0,\infty)$;
(ii) If $0 \leq p_1 \leq p_2$ *and* $b \geq -1$*, then* $z \mapsto \mathcal{U}^{\mathrm{k}}_{p_1,q}(z)\mathcal{U}^{\mathrm{k}}_{p_2,q}(z)/\mathcal{U}^{\mathrm{k}}_{\frac{p_1+p_2}{2},q}(z)$ *is decreasing on* $(0,\infty)$;
(iii) If $p+((q+1)/2)\mathrm{k} \geq 1/4$*, then* $z \mapsto \mathcal{U}^{\mathrm{k}}_{p,q}(z)/\exp(z^2)$ *is decreasing on* $(0,\infty)$;
(iv) If $\lambda \geq \nu > -\mathrm{k}$ *and* $2p+(q+1)\mathrm{k} > 0$*, then* $z \mapsto \mathcal{U}^{\mathrm{k}}_{p,q}(z)/{}_1\mathrm{F}_{1,\mathrm{k}}[(\lambda,\mathrm{k});(\nu,\mathrm{k});z]$ *is strictly decreasing on* $(0,\infty)$.

Proof. (i) Using the power series expansion of $\mathcal{U}^{\mathrm{k}}_{p,q}(z)$, we have

$$\frac{\mathcal{U}^{\mathrm{k}}_{p,q}(z)}{\mathcal{U}^{\mathrm{k}}_{p_1,q}(z)} = \frac{\sum\limits_{n=0}^{\infty}\sigma_n(p)\,z^{2n}}{\sum\limits_{n=0}^{\infty}\sigma(p_1)\,z^{2n}}.$$

In view of Lemma 1, it is enough to prove the monotonicity of

$$\mathrm{F}_n(z) = \frac{\sigma_n(p)}{\sigma_n(p_1)} = \frac{\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left(p_1+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)},$$

to get

$$\frac{\mathrm{F}_{n+1}(z)}{\mathrm{F}_n(z)} = \frac{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p_1+\frac{q+3}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+3}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)}.$$

Let

$$\zeta(p_1) = \frac{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p_1+\frac{q+3}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)}. \tag{25}$$

Differentiating (25) logarithmically with respect to p_1, it finds

$$\frac{\zeta'(p_1)}{\zeta(p_1)} = \Psi_{\mathrm{k}}\left(n\mathrm{k}+p_1+\frac{q+3}{2}\mathrm{k}\right) - \Psi_{\mathrm{k}}\left(n\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right).$$

Here, Ψ_{k} stands for the k-digamma function defined by

$$\Psi_{\mathrm{k}}(z) = \frac{\partial}{\partial z}\log\Gamma_{\mathrm{k}}(z) = \frac{\Gamma'_{\mathrm{k}}(z)}{\Gamma_{\mathrm{k}}(z)}.$$

Using the well-known representation for the k-digamma function

$$\Psi_{\mathrm{k}}(z)=\frac{\ln\mathrm{k}-\gamma}{\mathrm{k}}+\int_0^1\frac{\ell^{\mathrm{k}-1}-\ell^{z-1}}{1-\ell^{\mathrm{k}}}d\ell, \tag{26}$$

where γ denotes the Euler-Mascheroni constant given by the series

$$\gamma=\lim_{n\to\infty}\left(\sum_{n=1}^{\infty}\frac{1}{n}-\log(n)\right)=0.5772156649,$$

to find $\zeta'(p_1)\geq 0$ so that $\mathrm{F}_{n+1}(z)\geq\mathrm{F}_n(z)$, if $p_1\geq p$.

(ii) By using the Cauchy product of two power series, we have

$$\frac{\mathcal{U}^{\mathrm{k}}_{p_1,q}(z)\mathcal{U}^{\mathrm{k}}_{p_2,q}(z)}{\mathcal{U}^{\mathrm{k}}_{\frac{p_1+p_2}{2},q}(z)}=\frac{\sum\limits_{n=0}^{\infty}A_nz^{2n}}{\sum\limits_{n=0}^{\infty}B_nz^{2n}},$$

where

$$A_n=\sum_{\ell=0}^{n}\frac{\Gamma_{\mathrm{k}}\left(p_1+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(p_2+\frac{q+1}{2}\mathrm{k}\right)}{4^n\Gamma_{\mathrm{k}}\left(\ell\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left((n-\ell)\mathrm{k}+p_2+\frac{q+1}{2}\mathrm{k}\right)\ell!(n-\ell)!},$$

and

$$B_n=\sum_{\ell=0}^{n}\frac{\left[\Gamma_{\mathrm{k}}\left(\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\right]^2}{4^n\Gamma_{\mathrm{k}}\left(\ell\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left((n-\ell)\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\ell!(n-\ell)!}.$$

Suppose that

$$\mathrm{V}_\ell=\frac{\left[\Gamma_{\mathrm{k}}\left(\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\right]^2\Gamma_{\mathrm{k}}\left(\ell\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left((n-\ell)\mathrm{k}+p_2+\frac{q+1}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left(\ell\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left((n-\ell)\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(p_1+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(p_2+\frac{q+1}{2}\mathrm{k}\right)}.$$

Thus,

$$\begin{aligned}\frac{\mathrm{V}_{\ell+1}}{\mathrm{V}_\ell}&=\frac{\Gamma_{\mathrm{k}}\left((\ell+1)\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(\ell\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left((\ell+1)\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(\ell\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)}\\&\times\frac{\Gamma_{\mathrm{k}}\left((n-\ell-1)\mathrm{k}+p_2+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left((n-\ell)\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left((n-\ell-1)\mathrm{k}+\frac{p_1+p_2}{2}+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left((n-\ell)\mathrm{k}+p_1+\frac{q+1}{2}\mathrm{k}\right)}.\end{aligned}$$

Since $p_1\leq p_2$ and $q\geq-1$, it is easy to verify that $\mathrm{V}_{\ell+1}\leq\mathrm{V}_\ell$ which leads to the required result.

(iii) In the light of Lemma 1, the monotonicity of

$$\mathrm{W}_n=\frac{\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)}{4^n\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)},$$

must be tested. Since

$$\frac{\mathrm{W}_{n+1}}{\mathrm{W}_n}=\frac{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)}{4\Gamma_{\mathrm{k}}\left(n\mathrm{k}+\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)}=\frac{1}{4\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)}<1,$$

this leads to $\mathcal{U}^{\mathrm{k}}_{p,q}(z)/\exp(z^2)$ is decreasing on $(0,\infty)$.

(iv) Using (8) and the infinite series representation (7) with Lemma 1, it is enough to check the monotonicity of

$$\mathrm{X}_n=\frac{\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(\lambda\right)\Gamma_{\mathrm{k}}\left(n\mathrm{k}+\nu\right)}{4^n\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)\Gamma_{\mathrm{k}}\left(\nu\right)\Gamma_{\mathrm{k}}\left(n\mathrm{k}+\lambda\right)}.$$

Since

$$\frac{\mathrm{X}_{n+1}}{\mathrm{X}_n}=\frac{n\mathrm{k}+\nu}{4(n\mathrm{k}+\lambda)\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)}<1,$$

from the above assumptions, which implies to $z\mapsto\mathcal{U}^{\mathrm{k}}_{p,q}(z)/{}_1\mathrm{F}_{1,\mathrm{k}}[(\lambda,\mathrm{k});(\nu,\mathrm{k});z]$ is strictly decreasing on $(0,\infty)$ and consequently the proof is complete. □

Theorem 3. *Let* $\mathrm{k}>0,\ z>0$ *and* $2p+(q+1)\mathrm{k}>0$. *Then the following assertions hold:*

(i) The function $z\mapsto\mathcal{U}^{\mathrm{k}}_{p,q}(z)$ *is absolutely monotonic function on* $(0,\infty)$;

(ii) The function $\mathrm{Y}^{\mathrm{k}}_{p,q}(z):=\mathcal{U}^{\mathrm{k}}_{p,q}(\sqrt{z})-1$ *is super-additive on* $(0,\infty)$;

(iii) The function $p\mapsto\mathcal{U}^{\mathrm{k}}_{p,q}(z)$ *is decreasing on* $(0,\infty)$;

(iv) The function $p\mapsto\mathcal{U}^{\mathrm{k}}_{p,q}(z)$ *is log-convex on* $(0,\infty)$. *Furthermore,*

$$\left(\mathcal{U}^{\mathrm{k}}_{p,q}(z)\right)^2<\mathcal{U}^{\mathrm{k}}_{p-\mathrm{k},q}(z)\ \mathcal{U}^{\mathrm{k}}_{p+\mathrm{k},q}(z);$$

(v) The function $b\mapsto\mathcal{U}^{\mathrm{k}}_{p,q}(z)$ *is decreasing on* $(0,\infty)$;

(vi) The function $b\mapsto\mathcal{U}^{\mathrm{k}}_{p,q}(z)$ *is log-convex on* $(0,\infty)$. *Moreover,*

$$\left(\mathcal{U}^{\mathrm{k}}_{p,q}(z)\right)^2<\mathcal{U}^{\mathrm{k}}_{p,q-\mathrm{k}}(z)\ \mathcal{U}^{\mathrm{k}}_{p,q+\mathrm{k}}(z);$$

Proof. (i) Using the integral representation (4), we have

$$\mathcal{U}^{\mathrm{k}}_{p,q}(z)=\frac{2\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\int_0^1(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1}\cosh\left(\frac{\sqrt{r}\ell z}{\sqrt{\mathrm{k}}}\right)d\ell,$$

and so

$$\frac{\partial^n}{\partial z^n}\left(\mathcal{U}^{\mathrm{k}}_{p,q}(z)\right)=\frac{2\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)}{\sqrt{\pi\mathrm{k}}\Gamma_{\mathrm{k}}\left(p+\frac{q}{2}\mathrm{k}\right)}\int_0^1(1-\ell^2)^{\frac{p}{\mathrm{k}}+\frac{q}{2}-1}\frac{\partial^n}{\partial z^n}\cosh\left(\frac{\sqrt{r}\ell z}{\sqrt{\mathrm{k}}}\right)d\ell\geq 0,$$

This would lead to the absolute monotonicity of $\mathcal{U}^{\mathrm{k}}_{p,q}(z)$. Another way of proving the absolutely monotonicity of the prescribed function consists of using the fact that the power series of $\mathcal{U}^{\mathrm{k}}_{p,q}(z)$ has a nonnegative coefficient for $2p+(q+1)\mathrm{k}>0$ and $z>0$ (see [6]).

(ii) The function $\mathrm{Y}^{\mathrm{k}}_{p,q}(z)$ is super-additive on $(0,\infty)$ if $(\partial/\partial z)(\mathrm{Y}^{\mathrm{k}}_{p,q}(z)/z)\geq 0$, so

$$\frac{\partial}{\partial z}\left(\frac{\mathrm{Y}^{\mathrm{k}}_{p,q}(z)}{z}\right)=\frac{z\frac{\partial}{\partial z}(\mathcal{U}^{\mathrm{k}}_{p,q}(\sqrt{z}))-\left(\mathcal{U}^{\mathrm{k}}_{p,q}(\sqrt{z})-1\right)}{z^2},$$

and

$$z\frac{\partial}{\partial z}(\mathcal{U}^{\mathrm{k}}_{p,q}(\sqrt{z}))-\left(\mathcal{U}^{\mathrm{k}}_{p,q}(\sqrt{z})-1\right)=\sum_{n=1}^{\infty}\frac{(n-1)\,\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)z^{n-1}}{4^n\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\geq 0,$$

which implies to $\mathrm{Y}^{\mathrm{k}}_{p,q}(z):=\mathcal{U}^{\mathrm{k}}_{p,q}(\sqrt{z})-1$ is super-additive on $(0,\infty)$ for $2p+(q+1)\mathrm{k}>0$, $\mathrm{k}>0$ and $z>0$.

To complete the proof of (iii)-(vi), let us assume that

$$\Lambda^{\mathrm{k}}_{p,q}(z):=\frac{\Gamma_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)}{4^n\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}.\tag{27}$$

(iii) The proof follows by evaluating the logarithmic derivative of (27) for $\Lambda^{\mathrm{k}}_{p,q}(z)$ as:

$$\frac{\partial}{\partial p}\log\Lambda^{\mathrm{k}}_{p,q}(z)=\Psi_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)-\Psi_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right).$$

Using the representation (26), this implies that $p\mapsto\Lambda^{\mathrm{k}}_{p,q}(z)$ is decreasing on $(0,\infty)$. Since the infinite sum of decreasing functions is also decreasing, this leads to $p\mapsto\mathcal{U}^{\mathrm{k}}_{p,q}(z)$ is decreasing on $2p+(q+1)\mathrm{k}>0$ and $z>0$.

(iv) Since

$$\frac{\partial^2}{\partial p^2}\log\Lambda^{\mathrm{k}}_{p,q}(z)=\Psi'_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)-\Psi'_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right),$$

and by using the well-known representation:

$$\Psi'_{\mathrm{k}}(t)=\sum_{m=0}^{\infty}\frac{1}{(t+m\mathrm{k})^2},\ t\in\mathbb{R}\backslash\{0,-1,-2,...\},$$

we have

$$\begin{aligned}&\Psi'_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)-\Psi'_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)\\&=\sum_{m=0}^{\infty}\frac{(n\mathrm{k})^2+2n\mathrm{k}\left(m\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)}{\left(m\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)^2\left(n\mathrm{k}+m\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)^2}\geq 0,\end{aligned}$$

which implies that

$$\frac{\partial^2}{\partial p^2}\log\Lambda^{\mathrm{k}}_{p,q}(z)\geq 0,$$

for $2p+(q+1)\mathrm{k}>0$ and $z>0$, that is, the function $p\mapsto \Lambda^{\mathrm{k}}_{p,q}(z)$ is log-convex on $(0,\infty)$. Since the infinite sum of log-convex functions is log-convex too. This implies to $p\mapsto \mathcal{U}^{\mathrm{k}}_{p,q}(z)$ is log-convex on $(0,\infty)$ for $2p+(q+1)\mathrm{k}>0$ and $z>0$. On the other hand,

$$\log\mathcal{U}^{\mathrm{k}}_{p,q}(z)<\frac{1}{2}\Big(\log\mathcal{U}^{\mathrm{k}}_{p-\mathrm{k},q}(z)+\log\mathcal{U}^{\mathrm{k}}_{p+\mathrm{k},q}(z)\Big),$$

or equivalently $\big(\mathcal{U}^{\mathrm{k}}_{p,q}(z)\big)^2<\mathcal{U}^{\mathrm{k}}_{p-\mathrm{k},q}(z)\,\mathcal{U}^{\mathrm{k}}_{p+\mathrm{k},q}(z)$.

(v) From (27), we obtain

$$\frac{\partial}{\partial q}\log\Lambda^{\mathrm{k}}_{p,q}(z)=\frac{\mathrm{k}}{2}\Psi_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)-\frac{\mathrm{k}}{2}\Psi_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)\le 0, \qquad (28)$$

which implies to $q\mapsto \mathcal{U}^{\mathrm{k}}_{p,q}(z)$ is decreasing on $(0,\infty)$.

(vi) From (28), we have

$$\frac{\partial^2}{\partial q^2}\log\Lambda^{\mathrm{k}}_{p,q}(z)=\frac{\mathrm{k}^2}{4}\Psi'_{\mathrm{k}}\left(p+\frac{q+1}{2}\mathrm{k}\right)-\frac{\mathrm{k}^2}{4}\Psi'_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)\ge 0,$$

this leads to $q\mapsto \mathcal{U}^{\mathrm{k}}_{p,q}(z)$ is log-convex on $(0,\infty)$ which ends the proof of the last theorem. □

4. CONCLUSIONS

In the present investigation, interesting properties of the k-analogue of the generalized Bessel function of the first kind of order p, that is, $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ have been derived. We further have discussed the uniform convergence of $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$. Moreover, we have proved that the prescribed function is entire and found its growth order, type and Weierstrass factorization. Furthermore, the integral representation for $\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)$ is obtained using the representation for the k-beta functions. We further have proved that the specified function is a solution of a second-order differential equation that generalizes certain well-known differential equations for the classical Bessel functions. In addition, some interesting properties like the recurrence as well as differential relations, have been demonstrated and some of them have been used to establish some Turán type inequalities for it. Ultimately, we have studied the monotonicity and log-convexity of the normalized form of the modified k-Bessel function $\mathrm{T}^{\mathrm{k}}_{p,q,1}$ defined by $\mathrm{T}^{\mathrm{k}}_{p,q,1}(z)=i^{-\frac{p}{k}}\mathrm{U}^{\mathrm{k}}_{p,q,1}(iz)$ as well as the ratio of modified k-Bessel function and the exponential and k-hypergeometric functions where the leading concept of the proofs comes from the monotonicity of the ratio of two power series.

5. Appendix A. Generalized k-Bessel differential equation

Proof of Proposition 1

It is easy to observe from (8) that

$$z\frac{d}{dz}\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)=\sum_{n=0}^{\infty}\frac{(-r)^n\left(2n+\frac{p}{\mathrm{k}}\right)}{\Gamma_k\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}. \tag{29}$$

Differentiating (29) again with respect to z, it finds

$$\begin{aligned}
&z^2\frac{d^2\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}{dz^2}+z\frac{d\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)}{dz}=\sum_{n=0}^{\infty}\frac{(-r)^n\left(2n+\frac{p}{\mathrm{k}}\right)^2}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&=\sum_{n=0}^{\infty}\frac{(-r)^n\left(4n^2+4n\frac{p}{\mathrm{k}}+\frac{p^2}{\mathrm{k}^2}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&=\frac{p^2}{\mathrm{k}^2}\sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&+\sum_{n=0}^{\infty}\frac{4n(-r)^n\left(n+\frac{p}{\mathrm{k}}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&=\frac{p^2}{\mathrm{k}^2}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)+\sum_{n=0}^{\infty}\frac{4n(-r)^n\left(n+\frac{p}{\mathrm{k}}+\frac{q-1}{2}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&-(q-1)\sum_{n=0}^{\infty}\frac{2n(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&=\frac{p^2}{\mathrm{k}^2}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)+\frac{4}{\mathrm{k}}\sum_{n=1}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q-1}{2}\mathrm{k}\right)(n-1)!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&-(q-1)\sum_{n=0}^{\infty}\frac{(-r)^n\left(2n+\frac{p}{\mathrm{k}}-\frac{p}{\mathrm{k}}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&=\frac{p^2}{\mathrm{k}^2}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)+\frac{4}{\mathrm{k}}\sum_{n=0}^{\infty}\frac{(-r)^{n+1}}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+\mathrm{k}+p+\frac{q-1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&-(q-1)\,z\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'+\frac{p}{\mathrm{k}}(q-1)\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\\
&=\frac{p^2}{\mathrm{k}^2}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)-\frac{rz^2}{\mathrm{k}}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)-(q-1)\,z\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'+\frac{p}{\mathrm{k}}(q-1)\mathrm{U}^{\mathrm{k}}_{p,q,r}(z).
\end{aligned}$$

On simplification we easily arrive at the Eq. (16).

6. Appendix B. Recurrence and differential relations for generalized k-Bessel function

B.1. Proof of Proposition 2. (i) We shall evaluate the expression $\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)+\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)$ as follows:

$$
\begin{aligned}
&\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)+\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)=\sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q-1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&+\sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+3}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p+\mathrm{k}}{\mathrm{k}}}=\frac{1}{\Gamma_{\mathrm{k}}\left(p+\frac{q-1}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&+\sum_{n=1}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q-1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&+\sum_{n=1}^{\infty}\frac{(-r)^{n-1}}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)(n-1)!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&=\frac{1}{\Gamma_{\mathrm{k}}\left(p+\frac{q-1}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p-\mathrm{k}}{\mathrm{k}}}+\sum_{n=1}^{\infty}\frac{(-r)^n\left(n\mathrm{k}+p+\frac{q-1}{2}\mathrm{k}-\frac{n}{r}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&=\frac{1}{\Gamma_{\mathrm{k}}\left(p+\frac{q-1}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p-\mathrm{k}}{\mathrm{k}}}+\sum_{n=1}^{\infty}\frac{(-r)^n\left(p+\frac{q-1}{2}\mathrm{k}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&+\left(\mathrm{k}-\frac{1}{r}\right)\sum_{n=1}^{\infty}\frac{n(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&=\frac{1}{\Gamma_{\mathrm{k}}\left(p+\frac{q-1}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p-\mathrm{k}}{\mathrm{k}}}+\frac{2p+(q-1)\mathrm{k}}{z}\sum_{n=1}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\\
&+\left(\mathrm{k}-\frac{1}{r}\right)\sum_{n=1}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)(n-1)!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&=\frac{2}{z}\left(p+\frac{q-1}{2}\mathrm{k}\right)\\
&\times\left[\frac{1}{\Gamma_{\mathrm{k}}\left(p+\frac{q-1}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p}{\mathrm{k}}}+\sum_{n=1}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p}{\mathrm{k}}}\right]\\
&+(\mathrm{k}r-1)\sum_{n=1}^{\infty}\frac{(-1)^n r^{n-1}}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)(n-1)!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&=\frac{2}{z}\left(p+\frac{q-1}{2}\mathrm{k}\right)\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)+(1-\mathrm{k}r)\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z).
\end{aligned}
$$

After simple computations, we can arrive at (i). Secondly, to prove (ii), we shall aim to estimate the expression $\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)-\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)$ as follows:

$$\begin{aligned}
&\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)-\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)=\frac{1}{\Gamma_{\mathrm{k}}\left(p+\frac{q-1}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&+\sum_{n=1}^{\infty}\frac{(-r)^n\left(n\mathrm{k}+p+\frac{q-1}{2}\mathrm{k}+\frac{n}{r}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&=\frac{1}{\Gamma_{\mathrm{k}}\left(p+\frac{q-1}{2}\mathrm{k}\right)}\left(\frac{z}{2}\right)^{\frac{p-\mathrm{k}}{\mathrm{k}}}+\sum_{n=1}^{\infty}\frac{(-r)^n\left(n\mathrm{k}+\frac{p}{2}\right)}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&+\frac{p+(q-1)\,\mathrm{k}}{2}\sum_{n=1}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&-\sum_{n=1}^{\infty}\frac{(-r)^{n-1}}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p+\frac{q+1}{2}\mathrm{k}\right)(n-1)!}\left(\frac{z}{2}\right)^{2n+\frac{p-\mathrm{k}}{\mathrm{k}}}\\
&=\mathrm{k}\left(\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)'+\frac{p+(q-1)\mathrm{k}}{z}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)-\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z),
\end{aligned}$$

which implies to (ii) after some calculations. Moreover, to prove (iii), it is well-known that (i) can be expressed as

$$\mathrm{U}^{\mathrm{k}}_{p-\mathrm{k},q,r}(z)+\mathrm{k}r\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)=\frac{2p+(q-1)\mathrm{k}}{z}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z). \tag{30}$$

Multiplying both sides of (30) by $-r\mathrm{k}$ and replace p by $p+2\mathrm{k}$, then

$$-r\mathrm{k}\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)-\mathrm{k}^2r^2\mathrm{U}^{\mathrm{k}}_{p+\mathrm{k},q,r}(z)=\frac{2(p+2\mathrm{k})+(q-1)\mathrm{k}}{z}\mathrm{U}^{\mathrm{k}}_{p+2\mathrm{k},q,r}(z).$$

Again, multiplying both sides of (30) by $r^2\mathrm{k}^2$ and replace p by $p+4\mathrm{k}$, then

$$r^2\mathrm{k}^2\mathrm{W}^{\mathrm{k}}_{p+3\mathrm{k},q,r}(z)+\mathrm{k}^3r^3\mathrm{W}^{\mathrm{k}}_{p+5\mathrm{k},q,r}(z)=\frac{2(p+4\mathrm{k})+(q-1)\mathrm{k}}{z}\mathrm{W}^{\mathrm{k}}_{p+4\mathrm{k},q,r}(z).$$

Continuing in this manner and adding them to get (iii). By adding (i) and (iii), we get (iv). Combining (ii) and (iv), we get (v) which ends the proof.

B.2. Proof of Proposition 2. We shall prove (19) by applying the mathematical induction. For $m=1$, we have (18). Assuming that (19) is valid for $m=\upsilon$. We have to prove that it also holds for $m=\upsilon+1$ as follows:

$$\begin{aligned}
&\left(\frac{1}{z}\frac{d}{dz}\right)^{\upsilon+1}\left(z^{\frac{p}{\mathrm{k}}+q-1}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)=\left(\frac{1}{z}\frac{d}{dz}\right)\left(\frac{1}{z}\frac{d}{dz}\right)^{\upsilon}\left(z^{\frac{p}{\mathrm{k}}+q-1}\mathrm{U}^{\mathrm{k}}_{p,q,r}(z)\right)\\
&=\frac{1}{z}\frac{d}{dz}\left[\frac{1}{\mathrm{k}^{\upsilon}}z^{\frac{p}{\mathrm{k}}+q-(\upsilon+1)}\mathrm{U}^{\mathrm{k}}_{p-\upsilon\mathrm{k},q,r}(z)\right]\\
&=\frac{1}{z}\frac{d}{dz}\left[\frac{1}{\mathrm{k}^{\upsilon}}z^{\frac{p}{\mathrm{k}}+q-(\upsilon+1)}\sum_{n=0}^{\infty}\frac{(-r)^n}{\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p-\upsilon\mathrm{k}+\frac{q+1}{2}\mathrm{k}\right)n!}\left(\frac{z}{2}\right)^{2n+\frac{p-\upsilon\mathrm{k}}{\mathrm{k}}}\right]\\
&=\frac{1}{\mathrm{k}^{\upsilon}z}\sum_{n=0}^{\infty}\frac{(-r)^n\left(2n-2\upsilon+2\frac{p}{\mathrm{k}}+q-1\right)}{2^{2n+\frac{p-\upsilon\mathrm{k}}{\mathrm{k}}}\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p-\upsilon\mathrm{k}+\frac{q+1}{2}\mathrm{k}\right)n!}z^{2n-2\upsilon+2\frac{p}{\mathrm{k}}+q-2}\\
&=\frac{2}{\mathrm{k}^{\upsilon+1}z}\sum_{n=0}^{\infty}\frac{(-r)^n}{2^{2n+\frac{p-\upsilon\mathrm{k}}{\mathrm{k}}}\Gamma_{\mathrm{k}}\left(n\mathrm{k}+p-\upsilon\mathrm{k}+\frac{q-1}{2}\mathrm{k}\right)n!}z^{2n-2\upsilon+2\frac{p}{\mathrm{k}}+q-2}\\
&=\frac{1}{\mathrm{k}^{\upsilon+1}}z^{\frac{p}{\mathrm{k}}+q-(\upsilon+2)}\mathrm{U}^{\mathrm{k}}_{p-(\upsilon+1)\mathrm{k},q,r}(z).
\end{aligned}$$

Thus (19) is true for $m=\upsilon+1$ and consequently it holds for every natural number m. Similarly, we can prove relation (20).

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